Can simple renormalization theories describe the trapping of chaotic trajectories in mixed systems?

M. Weiss,[1,2] L. Hufnagel,[1] and R. Ketzmerick[1,3]
[1]*Max-Planck-Institut für Strömungsforschung and Institut für Nichtlineare Dynamik der Universität Göttingen, Bunsenstrasse 10, 37073 Göttingen, Germany*
[2]*EMBL, Meyerhofstrasse 1, 69117 Heidelberg, Germany*
[3]*Institut für Theoretische Physik, Technische Universität Dresden, 01062 Dresden, Germany*
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We investigate the relation between the chaotic dynamics and the hierarchical phase-space structure of the standard map as an example for generic Hamiltonian systems with a mixed phase space. We demonstrate that even in ideal situations when the phase-space structure is dominated by a single scaling, the long-time dynamics is *not* dominated by this scaling. This has consequences for the power-law decay of correlations and Poincaré recurrences.

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I. INTRODUCTION

Generic Hamiltonian systems are neither integrable nor chaotic [1], but rather exhibit a mixed phase space, where regular and chaotic regions coexist. Each island of regular motion is surrounded by infinitely many chains of smaller islands. As the same holds for any of these smaller islands a very complex hierarchical phase-space structure is found for generic Hamiltonian systems, which is well understood [2] and nowadays appears in textbooks on classical mechanics [3].

The dynamical properties of these systems, however, are still poorly understood. The most fundamental statistical quantity for characterizing the dynamics is the decay of correlations in time. It determines transport properties and is directly related to the distribution of Poincaré recurrences $P(t)$, which is the probability to return to a given region in phase space with a recurrence time larger than t. This probability decays on average like a power law [4]

$$P(t) \sim t^{-\gamma}, \tag{1}$$

due to the trapping of chaotic trajectories in the hierarchically structured vicinity of islands of regular motion. The power-law decay is a universal property of Hamiltonian systems. It has dramatic consequences for transport [5] (anomalous diffusion) and quantum mechanical properties [6] (conductance fluctuations and eigenfunctions), which sensitively depend on the value of γ. The exponent γ, as determined from finite time numerical experiments, seems to be nonuniversal, varies with system and parameter, and typically ranges between 1 and 2.5 [4–6]. It is a fundamental question of Hamiltonian chaos, how the exponent γ of the dynamics is related to the structure of the hierarchical phase space.

Recently, it was argued by Chirikov and Shepelyansky that for asymptotically large times the exponent is independent of the specific system and parameter and is given by the universal exponent $\gamma=3$ [7]. Their arguments are based on the universal presence of critical tori in phase space and are supported by a numerical investigation of the standard map (kicked rotor),

$$q_{n+1} = q_n + p_n \bmod 2\pi, \quad p_{n+1} = p_n + K \sin q_{n+1}, \tag{2}$$

at kicking strength $K = K_c = 0.971\,635\,406\,31$. At this parameter value, the golden torus is critical, i.e., it can be destroyed by an arbitrarily small perturbation. The self-similar vicinity of the critical golden torus [see Figs. 1(a,b)] has been studied using renormalization methods [8] and the asymptotic value $\gamma=3$ for the power-law decay of $P(t)$ was predicted long time ago [9,10]. The fact that it has never been observed led to the speculation that the universal decay should appear for larger times [11]. In Ref. [7], the onset of this decay was estimated by a numerical approach.

In addition to the sticking of trajectories in the vicinity of critical tori, the trapping of trajectories in island-around-island structures has been studied [12,13]. Zaslavsky *et al.* [13] showed that for the kicked rotor at $K = K^* = 6.908\,745$, the phase space possesses an island-around-island structure of sequence $3-8-8-8\cdots$ [see Fig. 3(a)]. They used this self-similarity to derive the trapping exponent $\gamma=2.25$ by renormalization arguments [14], which was recently supported numerically [15].

In fact, these renormalization approaches for single self-similar phase-space structures can be considered as special cases of the more general phase-space model by Meiss and Ott [16] for the trapping in the neighborhood of regular islands [17]. In this binary tree model a chaotic trajectory can at any stage of the tree either go to a boundary circle (level scaling) or to the island-around-island structure (class scaling). The universal coexistence of the two routes of renormalization at any stage led to the exponent $\gamma=1.96$ [16]. In contrast, the recent findings claim that just *one* of these scalings is relevant for the trapping mechanism: While in Ref. [7] it is argued that in the case of a typical border invariant curve (and in particular for $K = K_c$) the level scaling should

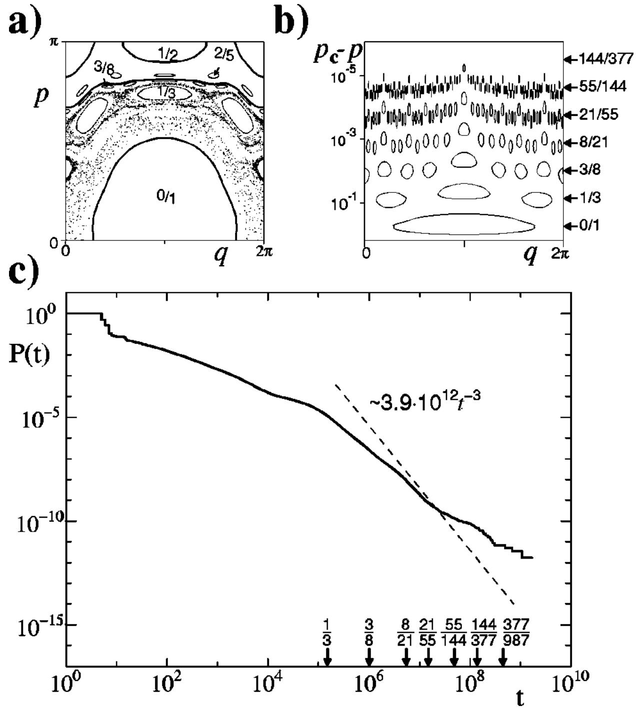

FIG. 1. (a) Phase space of the symmetrized standard map at $K=K_c$. As the phase space is symmetric about $p=\pi$ only the lower half of one unit cell is shown. (b) By stretching the phase space in p direction according to the distance $p_c(q)-p$ to the critical golden torus $p_c(q)$, the self-similar phase-space structure in its vicinity is visualized. The winding numbers of the principal resonances approaching the golden torus are shown on the right. (c) Poincaré recurrences $P(t)$ (solid) for the standard map with $K=K_c$, deviating from the prediction of Ref. [7] (dashed line). The winding numbers above each arrow specify the hierarchy level that a trajectory with the indicated trapping time should reach according to the renormalization prediction of Ref. [7].

dominate, in Ref. [13] it is claimed that for $K=K^*$ the class scaling describes the trapping mechanism.

In order to clarify these contradictions and to check whether it is sufficient to describe the dynamics by the scaling of a single structure, we numerically investigate $P(t)$ for the kicked rotor at K_c and K^* for very large times. We find considerable deviations from the predictions of the renormalization theories that rely only on the scaling of a single structure. The Poincaré recurrences $P(t)$ for $K=K_c$ have already been published in a short note [18]. In addition, we analyze where chaotic trajectories are trapped in phase space, which allows us to understand these deviations. For large times the majority of trajectories is not trapped in those phase-space regions that are described by the simple renormalization theories. This indicates that the decay of $P(t)$ cannot be captured by solely inspecting a single structure, be it level or class scaling. In particular for $K=K_c$, the self-similar vicinity of the critical golden torus does not dominate the trapping mechanism for large times and thus even in this ideal situation the proposed universal exponent $\gamma=3$ is not found. For $K=K^*$, the majority of long trapped trajectories does not follow the island-around-island structure, and we find a smaller exponent than predicted. Although the phase-space structure in both cases is dominated by a single scaling, our analysis shows that it does not necessarily dominate the dynamics.

II. THE CASE $K=K_C$

A. Statistics of Poincaré recurrences

The standard map as defined by Eq. (2) has a 2π-periodic phase space in p direction and for $K=K_c$ the dynamics is bounded in p direction by the golden torus, which is critical [8] [Figs. 1(a,b)]. The route towards the critical golden torus is determined by the principal resonances given by the approximants of the golden mean $\sigma=(\sqrt{5}-1)/2$ and the scaling has been analyzed in detail [8]. The dynamics along this route was described by a Markov chain leading to $\gamma=3.05$ [9] and alternatively via the scaling of the local diffusion rate leading to $\gamma=3$ [10].

In order to check the prediction for $P(t)$, we started several long trajectories initially located near the unstable fixed point $(q,p)=(0,0)$. We measured the times τ for which an orbit stays close to the critical torus by monitoring successive crossings of the line $p=0$, as was also done in Ref. [7]. The total computer time corresponds to 15×10^{12} iterations of the standard map, which came from 15 trajectories of length 10^{12}. We have checked if our statistical data for large times are sensitive to the unavoidable finite numerical precision, by comparing data for double (≈16 significant digits) and quadruple (≈32 digits) precision [19]. We found no difference and present a combination of both data sets in Fig. 1(c).

In Fig. 1(c), we compare our numerical findings for $P(t)$ with the prediction $P(t>10^7)=3.9\times10^{12}t^{-3}$ as extracted from Ref. [7]. The predicted power law is not compatible with our data, even though we are in the time regime $t>10^7$, where it should be observable according to Ref. [7]. For $10^5\leqslant t\leqslant10^9$ we rather see an exponent $\gamma\approx1.9$. We also studied Poincaré recurrences for trajectories approaching the critical torus from the other side in the same way as in Ref. [7]. In the range from $10^8<t<10^9$ we find a very slow decay ($\gamma\approx1.2$), so that $P(t=10^9)$ is more than three orders of magnitude bigger than the prediction $P(t)=4\times10^{13}t^{-3}$ from Ref. [7] (see Fig. 1 in Ref. [18]).

B. Why renormalization fails

The failure of the renormalization theories in describing the decay of $P(t)$ may be due to at least two reasons: (i) the renormalization theory correctly describes the trapping along the considered phase-space structures, but the neglected structures give more important contributions to the trapping time statistics; (ii) the trapping is dominated by the considered phase-space regions but the renormalization fails to describe the associated dynamics correctly. This could be the case if, for example, the mixing time at each level is bigger than the exit time to the next level, making the dynamics between the levels non-Markovian.

In order to check what causes the deviations of $P(t)$ from the predictions, we calculate the density in phase space for

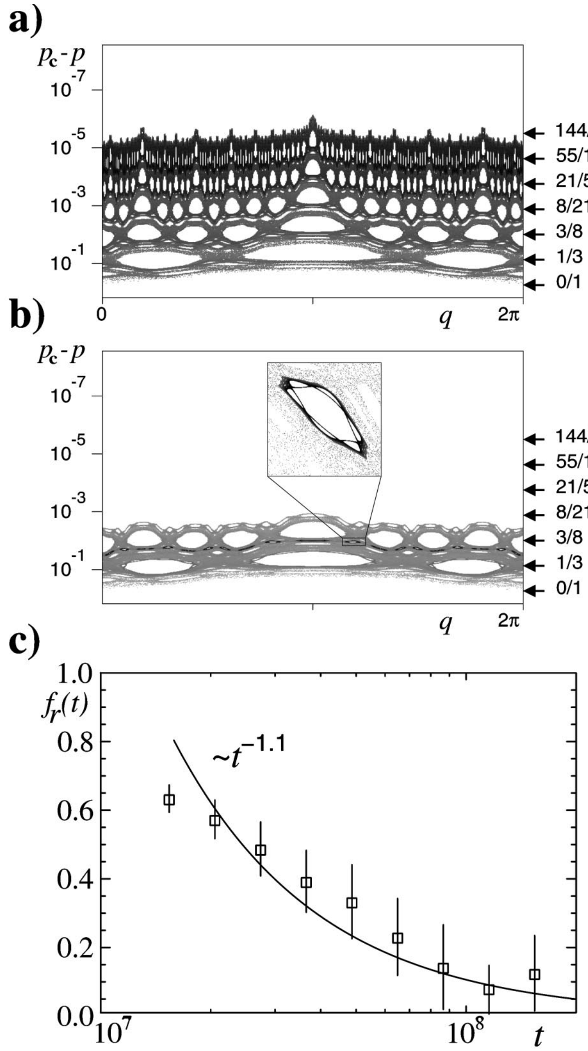

FIG. 2. (a) Phase-space density of a trajectory with trapping time $t \approx 5 \times 10^7$ for the symmetrized standard map at $K = K_c$ plotted in logarithmic style as in Fig. 1(b). The density is determined on a 250×250 grid, gray shadings are on a logarithmic scale. The trajectory follows the self-similar phase-space structure given by the principal resonances approaching the golden torus. (b) A representative counterexample, which is not trapped in the hierarchy of principal resonances as predicted by the renormalization approach, although it has the same trapping time $t \approx 5 \times 10^7$. Instead, the trajectory is trapped around a nonprincipal resonance (inset). (c) The fraction $f_r(t)$ of trajectories with trapping time t following the route of renormalization. The decay shows that the main contributions to $P(t)$ do not arise from trajectories that are trapped in the phase-space structure given by the class scaling of the principal resonances.

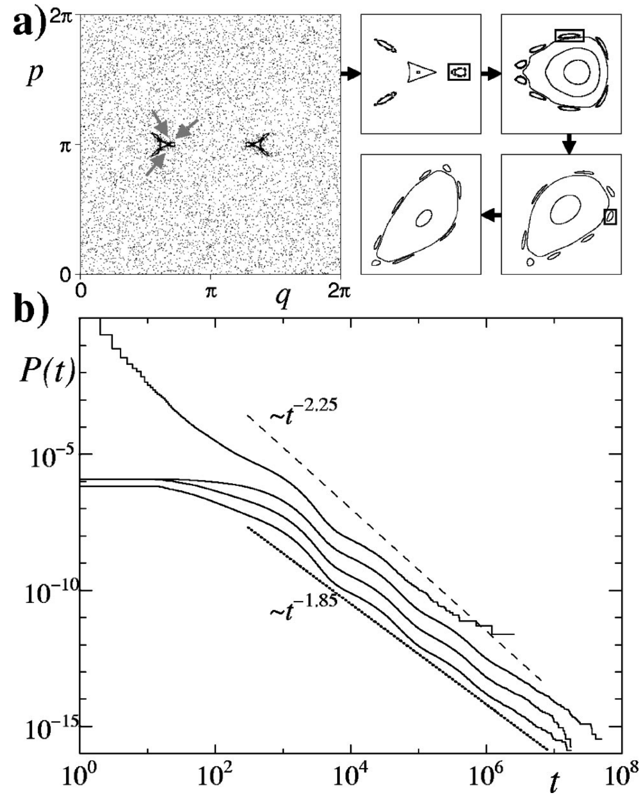

FIG. 3. (a) Phase space of the symmetrized standard map at $K = K^*$ with successive magnifications showing the island-around-island structure. (b) Poincaré recurrences $P(t)$ (solid) for the standard map with $K = K^*$ for various initial conditions (vertically shifted for better comparison). The upper curve shows the recurrences for trajectories started randomly on the line $q = \pi$. The other curves belong to trajectories started randomly in small boxes at three different positions close to the island as indicated by the arrows in (a). All four curves show the same power-law behavior, including the log-periodic oscillations, and deviate from the prediction $\gamma = 2.25$ from Ref. [13] (dashed).

trajectories that are trapped for long times. Two examples are shown in Figs. 2(a,b). Figure 2(a) shows a trajectory of length $t \approx 5 \times 10^7$ that follows the route to the critical golden torus up to the principal resonance with winding number 55/144. This is consistent with the renormalization theory according to the data presented in Ref. [7] [see arrows in Fig. 1(c)]. In contrast, the trajectory shown in Fig. 2(b), although trapped for the same time, approaches the critical torus only up to the resonance 3/8 and is predominantly trapped around a nonprincipal resonance [inset in Fig. 2(b)]. The contribution to $P(t)$ of such trajectories is not captured by the renormalization theory. In order to quantify their influence we determine from all trajectories with trapping time t the fraction $f_r(t)$ that follows the route of renormalization. If $f_r(t)$ approaches unity for large times, the asymptotic trapping is dominated by trajectories that follow the route of renormalization. If $f_r(t)$ decays to zero, however, other phase-space regions dominate the asymptotic trapping.

Numerically, we determine $f_r(t)$ by considering trajectories with trapping times in an interval around t. We classify these trajectories, as was done for the examples in Figs. 2(a,b), according to their phase-space densities: For a trajectory with trapping time t the renormalization theory together with the numerical findings in Ref. [7] predict that it should approach the golden torus up to a certain hierarchy level characterized by its winding number [arrows in Fig. 1(c)]. When the trajectory reaches this level or the one before, we classify it as following the route of renormalization and it contributes to $f_r(t)$. We find that $f_r(t)$ decays for $t > 2 \times 10^7$. While at $t = 1.5 \times 10^6$ more than 60% of all trajectories follow the route of renormalization, at $t\approx 10^8$ only 10% do so [Fig. 2(c)]. This indicates that for large times the majority of trajectories does not follow the route of renormalization towards the golden torus given by the principal resonances, but are trapped around nonprincipal resonances [like the example shown in Fig. 2(b)]. Since the renormalization theories that lead to $\gamma\approx 3$ consider only the trapping around principal resonances they miss for increasing time more and more trajectories that carry the long-time behavior of $P(t)$. In view of that, the renormalization theory is not applicable for predicting the decay of $P(t)$.

Nevertheless, the renormalization theories correctly describe the contribution to $P(t)$ due to the trapping around principal resonances, as can be seen as follows: From the ratio of the predicted $P(t)\sim t^{-3}$ for the trajectories trapped in the self-similar phase-space structure and the observed $P(t)\sim t^{-1.9}$, we expect the fraction f_r to decay as $f_r(t)\sim t^{-1.1}$ for large times. This is consistent with our numerical data in Fig. 2(c). We thus find from our numerical analysis that the failure of the renormalization approach is due to reason (i) mentioned at the beginning of this section and not due to reason (ii).

We find that the majority of trajectories is trapped around nonprincipal resonances, which is in agreement with the binary tree model [16]. This is in contrast to the conclusions of Ref. [7] that are based on the computation of exit times from the vicinity of unstable fixed points of principal resonances. The analysis of the mean exit time of a phase-space region as well as the investigation of the local diffusion rates in this region [20] can only give information about trajectories trapped in the considered region. While the mean exit time of a region in phase space determines the time when this region contributes to $P(t)$, it cannot tell how important the contribution is for the global trapping mechanism.

III. THE CASE $K=K^*$

A. Statistics of Poincaré recurrences

We now carry out the same analysis for $K=K^*$, where the phase space consists of two small accelerator modes embedded in an otherwise chaotic phase space [Fig. 3(a)]. Each accelerator mode shows an island-around-island structure of sequence $3-8-8-8\cdots$ [13]. This scaling was used to predict the exponent $\gamma=2.25$ [13].

Whenever a chaotic trajectory is trapped to one of the island structures, it follows the dynamics of the accelerator mode and jumps to the neighboring unit cell in p direction. The trapping time τ of a trajectory is thus the time it jumps one unit cell per iteration in the same direction. We determine the probability $\tilde{P}(t)$ of being trapped longer than a time t by starting trajectories randomly placed on a line $q=\pi$ away from the accelerator modes [upper curve in Fig. 3(b)]. From the set of trapping times τ one determines the fraction $\tilde{P}(t)$ of orbits with $\tau\geqslant t$. This quantity decays with the same power-law exponent as the Poincaré recurrences $P(t)$ and was chosen for numerical convenience. In order to increase

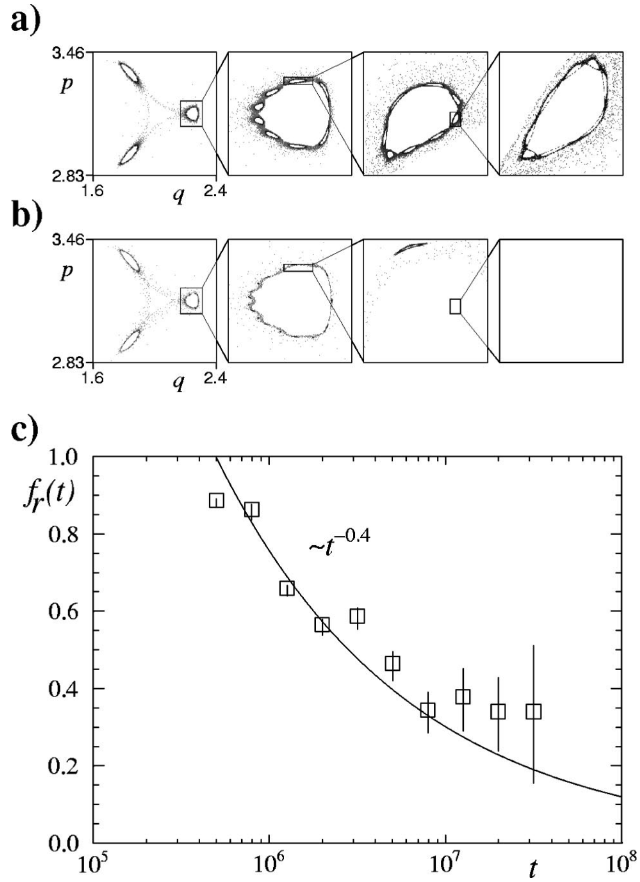

FIG. 4. (a) Phase-space density of a trajectory for the symmetrized standard map at $K=K^*$ with trapping time $t\approx 2\times 10^6$. The density is determined on a 250×250 grid and gray shadings are on a logarithmic scale. The trajectory follows the island-around-island structure. (b) A representative counterexample with the same trapping time $t\approx 2\times 10^6$, which is not trapped in the island-around-island hierarchy used for the renormalization approach, but around an island chain of period $3\times 17=51$ surrounding the period 3 islands. (c) The fraction $f_r(t)$ of trajectories with trapping time t, which follow the island-around-island hierarchy. The decay shows that the main contributions to $P(t)$ do not arise from the route of renormalization.

statistics for large times, we have started randomly placed trajectories in three different small boxes close to the accelerator mode in positive direction [arrows in Fig. 3(a)]. In principle, the exponent of the asymptotic decay of $P(t)$ might depend on the initial box, in particular, if it is chosen too close to the island. We find, however, that this is not the case for our choices, as all four curves show the same behavior (including the log-periodic oscillations) for times $t>2\times 10^3$. The total computer time corresponds to 8×10^{12} iterations of the standard map, with about 20 trajectories of length 10^{11} started in each of the four ensembles. Averaged over the log-periodic fluctuations we find a power-law decay of $P(t)$ with $\gamma=1.85$ [Fig. 3(b)], which is not compatible with the renormalization prediction $\gamma=2.25$ [13].

B. Why renormalization fails

In order to clarify the contradiction between the predicted and the numerically observed $P(t)$, we investigate the phase-space densities of individual trajectories. In Figs. 4(a,b), we show the phase-space densities of two long trajectories. Although both trajectories have the same trapping time, only the trajectory in Fig. 4(a) follows the island-around-island structure, while the trajectory shown in Fig. 4(b) is trapped around another chain of islands (see top of third box).

We have calculated the fraction $f_r(t)$ of trajectories that follow the route of renormalization [Fig. 4(c)]. At $t = 5\times10^5$ more than 80% of all the trajectories are consistent with the renormalization prediction, while at $t\approx10^7$ this fraction is decreased to 30%. The decay is consistent with the estimate $f_r(t)\sim t^{-2.25}/t^{-1.85}\sim t^{-0.4}$, i.e., the ratio of the predicted $P(t)\sim t^{-2.25}$ and the observed $P(t)\sim t^{-1.85}$. We note that the difference between the predicted and the measured value for γ is not as big as for $K=K_c$. Therefore, the decay of $f_r(t)$ is less strong and suffers from statistical fluctuations. Still, the majority of trajectories is not following the dominant island-around-island structure. This indicates why the renormalization theory for the island-around-island structure is not capable of explaining $P(t)$. It should be noted that this difference is not caused by the finite precision of K^* which eventually leads to a breakdown of the sequence $3-8-8-8-\cdots$ on very small scales. We thus find for $K=K^*$ that it does not seem to be a valid assumption that the route of renormalization dominates the contributions of $P(t)$.

IV. CONCLUSION

In conclusion, our analysis shows that even if the phase-space structure is dominated by a single scaling (level or class), it is not sufficient to describe the trapping mechanism of chaotic trajectories by only this scaling, as was recently claimed in the literature. We find that additional island structures may dominate the trapping mechanism for large times and thus affect the power-law decay of $P(t)$.

Our analysis supports qualitatively the tree model by Meiss and Ott [16], which allows for the coexistence of two routes of renormalization at any stage in the phase-space hierarchy, leading to $\gamma=1.96$. Given the fluctuations in $P(t)$, which render a precise determination of the exponents difficult, our numerical findings are consistent with this prediction. If for larger times our somewhat smaller numbers were verified, this would be an indication that more island families have to be included in the statistical description of the dynamics, as this yields smaller exponents [16].

It is thus necessary to examine the trapping of chaotic trajectories in more detail and it remains an open question if there exists a universal asymptotic exponent for the trapping of chaotic trajectories in Hamiltonian systems.

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[19] The unavoidable finite numerical precision may affect the data in two ways: (i) The roundoff errors increase exponentially and thus after a few iterations the numerically calculated trajectory deviates significantly from the correct trajectory. This effect is known from fully chaotic systems, where the shadowing theorem bridges the way for numerical simulations. (ii) For mixed systems the scaling properties of the hierarchical phase space lead to very fine structures. If these structures (e.g., turnstiles of cantori) are smaller than the numerical precision, they cannot be resolved. The influence of the latter effect can be checked by increasing the precision.

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